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Optimal paths and universality

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Abstract. Optimal paths in disordered systems are studied using two different models interpolating between weak and infinitely strong disorder. In one case, exact numerical methods are used to study the optimal path in a two-dimensional square lattice whereas a renormalization-group analysis is employed on hierarchical lattices in the other. The scaling behaviour is monitored as a function of parameters that tune the strength of the disorder. Two distinct scenarios are provided by the models: in the first, fractal behaviour occurs abruptly as soon as the disorder widens, while in the other it emerges as a limiting case of a self-affine regime.

The effect of disorder on critical phenomena is a subject of considerable interest. Well studied problems in this context are the geometry of domain walls in weakly disordered ferromagnets [1], directed polymers in random media [2] and the dynamics of growing interfaces governed by the Kardar–Parisi–Zhang (KPZ) equation [3]. Strikingly, these problems are related to each other in two dimensions via the Burgers equation [4]. Unlike the weak disorder case [2], there are many physical situations, for example, transport in amorphous semiconductors at low temperatures [5], electrical conduction and fluid flow in porous rocks [6] and the magnetic properties of doped semiconductors [7], in which the physical attributes have a broad distribution. An approximation that is operationally useful in such cases is to assume that the distribution is so wide that the sum of several variables chosen from the distribution is simply equal to the largest variable. This strong disorder limit can also be realized in a spin model in which the coupling magnitudes scale nonlinearly with volume [8,9]. Recently, it was shown [9] that, in this limit, the self-affine domain walls become fractally rough with significant overhanging configurations thus leading to a new universality class.

In this paper we address the issue of the crossover between the limits of weak and strong disorder. We show that, depending on the mechanism of tuning of the disorder, one obtains two distinct crossover scenarios: the first in which the strong disorder limit appears as soon as the disorder widens and the second in which it emerges as a limiting case of a self-affine regime.

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We first describe an optimization problem that is exactly equivalent to the domain wall problem. Consider a lattice made up of bonds of random strength. For concreteness, let us assume that the strength of the bond is equal to the time taken to traverse it. We seek the optimal path that minimizes the total travel time from one point to another. In the weak disorder limit the overhangs (or backward going segments) are not significant and the path is known to be self-affine [4]. The total path length l spanning a distance L scales as $l \sim L^{D_f}$, with $D_f = 1$, whereas the width of the path (or the magnitude of the transverse excursions), W, scales as $W \sim L^{\chi}$. Further, the RMS fluctuations in the travel time distribution scale as $\Delta E \sim L^{\omega}$ with $\chi = (1 + \omega)/2$. In $D = 2$, it is known exactly [1] that $\chi = \frac{2}{3}$ and $\omega = \frac{1}{3}$. In the strong disorder limit, the traversal time-space has an ultrametric structure. The minimum time for travelling from A to B (the path between A and B and all subpaths are optimally chosen) $C(A, B)$ satisfies the relation $C(A, B) \leqslant \mathrm{Max}(C(A, X), C(X, B))$ for any arbitrary X. In this limit, it was shown numerically [9] that $D_f \approx 1.2$ and $\chi \approx 1$ in $D = 2$. The optimal path is now self-similar.

We now turn to our numerical analysis in $D = 2$. We treat a random bond Ising model at $T = 0$ (this is equivalent to the travel time optimization problem, but on a dual lattice) on a square lattice with the Hamiltonian $H = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j$ where nearest-neighbour spins σ_i and σ_j are coupled by an exchange coupling J_{ij}. In order to force an interface we impose antiperiodic boundary conditions in one direction and periodic boundary conditions in the other. Provided that the J_{ij}'s are all ferromagnetic, this optimization problem maps directly onto the problem of finding the maximum flow in a network with capacity constraints on the edge [10]. In order to obtain the exact ground state, we used a max-flow algorithm devised by Goldberg [11], which has a running time of $O(n^3)$ for a graph with n connections, while requiring less storage than other parallel max-flow methods. For a given distribution of random bonds, we calculated the optimal interfacial position and measured l and the RMS W. We used lattices of size $L \times L$ with L varying in size from 8 to 96 and we averaged our results over 10 000 realizations for the smaller systems and 1000 for the largest.

The exchange couplings were chosen to be

$$J_{ij} = y\left(c^{L^2}\right)^{y-\frac{1}{2}} \tag{1}$$

where y is a random number chosen uniformly in the interval (0, 1) and $c \geqslant 1$ is a parameter that controls the strength of the disorder. Figure 1 shows typical interfaces generated in a $L = 96$ system for $c = 1$, 1.01 and 1.1. For $c = 1$, the interface is self-affine. For any $c > 1$, the interface crosses over from self-affine to fractal at sufficiently long length scales

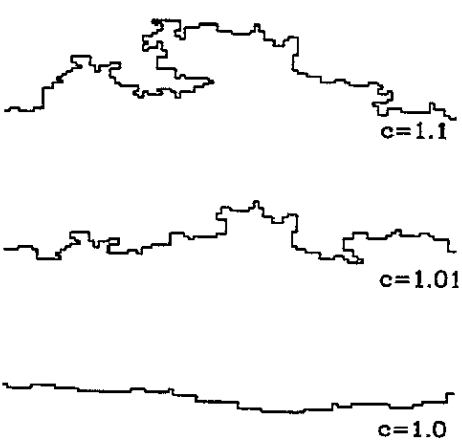

Figure 1. Domain walls in a two-dimensional random Ising ferromagnet ($L = 96$) with the exchange couplings given by (1) for the three values of c indicated.

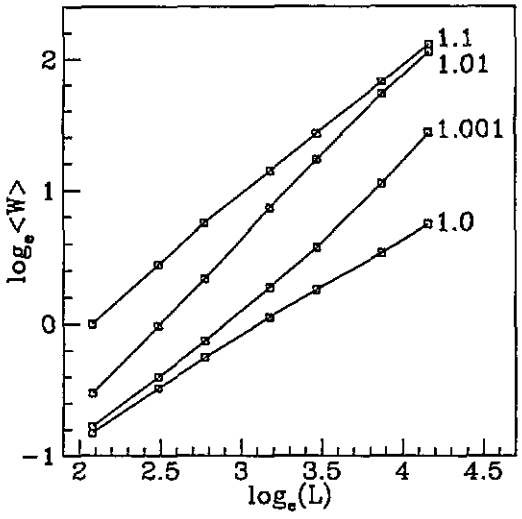

Figure 2. Average lateral width as a function of the linear system size in the two-dimensional Ising ferromagnet for the values of c as shown.

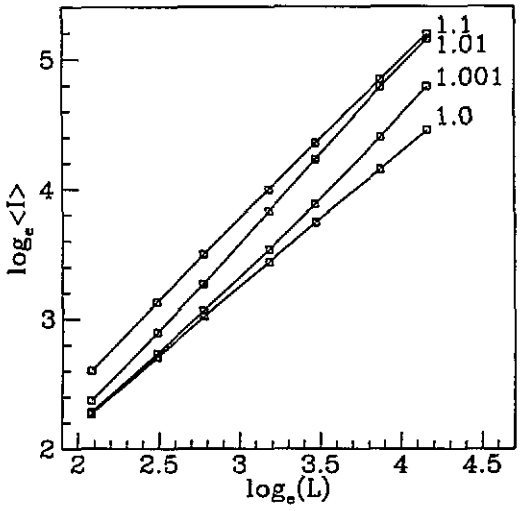

Figure 3. Average interface length, corresponding to the data of figure 2.

with the crossover occurring earlier for larger values of c (figures 2 and 3). In the weak disorder limit ($c = 1$), $\chi = 0.66 \pm 0.01$ in accord with the exact results [1], whereas at $c = 1.1$, $D_f = 1.21 \pm 0.03$ in agreement with the results of [9]. Note that, in the above analysis, the random distribution of bond strengths has been chosen to be dependent on the size of the system.

In order to obtain more insight into this behaviour, we turn to an analysis of hierarchical lattices [12] shown in figure 4. The hierarchical lattices are constructed in an iterative fashion by replacing each bond of the unit by the full unit in each step of the recursion. Each of the bonds represents both the traversal time y and the effective path length, l. In practice, a pool of typically 10^5 bonds is constructed. m members of the pool (the values of m are shown in figure 4) are picked at random and combined to give one new member of the new pool. The process is repeated until a new pool of 10^5 bonds is produced. (The pool size is large enough so that correlations are not significant for the number of iterations considered.) This corresponds to one iteration. For the starting pool, the traversal time, y_i, is picked

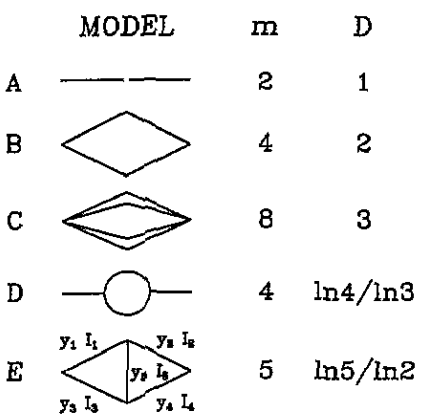

Figure 4. Hierarchical lattices considered in this study. System D was introduced in [15] to model the backbone of a percolation cluster. D represents the effective dimensionality of the hierarchical lattice.

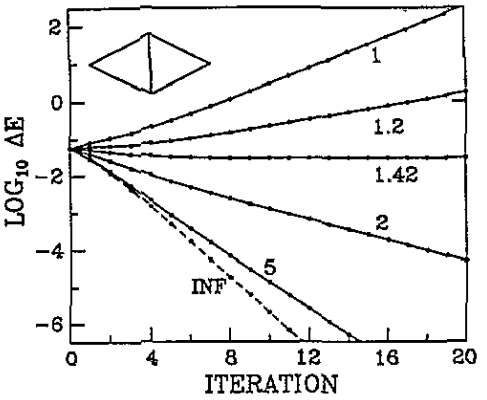

Figure 5. Variation of ΔE as a function of the iteration index for the hierarchical lattice E. The values of α are indicated on the figure.

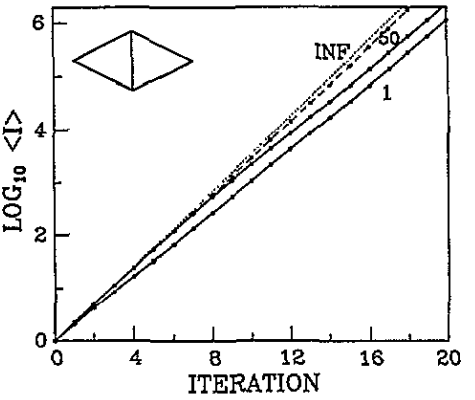

Figure 6. Variation of the average path length for $\alpha = 1, 50$ and ∞. The latter is indicated by the broken line corresponding to an exponent of 1.15. The dotted line corresponds to the analytically calculated exponent $D_\infty = 1.18$.

randomly and uniformly from the interval 0–1. The corresponding path length is 1. The total travel time for traversing a 1D sequence of bonds characterized by $y_1, y_2, \ldots, y_p$ is taken to be

$$\tilde{y} = \left[\sum_i^p {y_i}^\alpha\right]^{1/\alpha} . \tag{2}$$

Note that when $\alpha = 1$, the case considered in [13] for model B, this leads to $\tilde{y} = \sum_{i=1}^p y_i$, whereas for $\alpha = \infty$, $\tilde{y} = \max[y_1, y_2, \ldots, y_p]$. Thus the parameter α provides a convenient way for tuning from the usual definition to the strong disorder limit. For a given hierarchical model, all possible paths are considered and the effective travel times are calculated and the minimum overall time is selected. In cases, where there is a tie (which is common when $\alpha = \infty$), it is broken by ensuring that all sub-paths are optimal as well. After the best path is determined, the length of that particular path is recorded. Thus, for model E (figure 4), an entry of the new pool y is given by

$$y = \mathrm{Min}[(y_1^\alpha + y_2^\alpha)^{1/\alpha}, (y_1^\alpha + y_5^\alpha + y_4^\alpha)^{1/\alpha}, (y_3^\alpha + y_4^\alpha)^{1/\alpha}, (y_3^\alpha + y_5^\alpha + y_2^\alpha)^{1/\alpha}] \tag{3}$$

and $I = I_1 + I_2$, or $I = I_1 + I_5 + I_4$, or $I = I_3 + I_4$, or $I = I_3 + I_5 + I_2$ depending

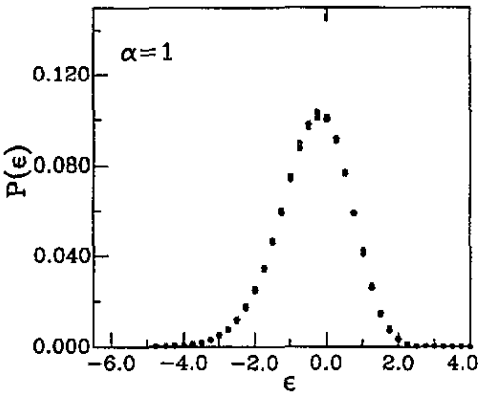

Figure 7. The fixed probability distribution of ϵ for $\alpha = 1$. ϵ is defined as $(y - \langle y \rangle)/\sigma_y$, where σ_y is the dispersion of y. The data points correspond to iterations 12, 16 and 20.

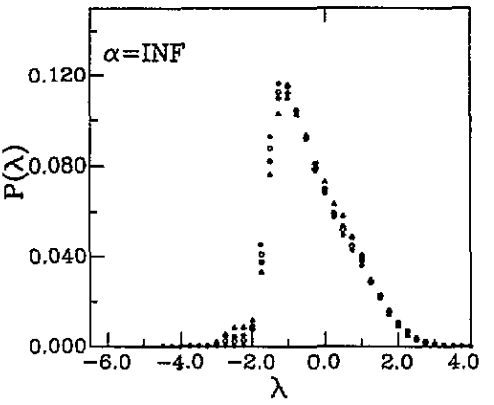

Figure 8. The fixed probability distribution at $\alpha = \infty$ for $\lambda = (I - \langle I \rangle)/\sigma_I$, where σ_I is the dispersion of I. The data points correspond to iterations 12, 16, 18 and 20.

Table 1. Exponents ω and D_f obtained for several hierarchical lattices. The analytic predictions for ω ($\alpha = \infty$) based on a connection to directed percolation (see the text) are also shown.

	ω				D_f		
System	$\alpha = 1$	$\alpha = 1.2$	$\alpha = \infty$	$-1/\nu_\parallel$	$\alpha = 1$	$\alpha = 1.2$	$\alpha = \infty$
A	0.5 ± 0.01	0.333 ± 0.003	-1.0 ± 0.06	-1	1	1	1
B	0.30 ± 0.01	0.13 ± 0.01	-0.60 ± 0.01	-0.612	1	1	1
C	0.21 ± 0.01	0.04 ± 0.01	-0.81 ± 0.04	-0.815	1	1	1
D	0.45 ± 0.02	0.29 ± 0.01	-0.63 ± 0.06	-0.631	1	1	1
E	0.30 ± 0.01	0.13 ± 0.01	-0.69 ± 0.01	-0.700	1	1	1.15 ± 0.003

on which term is the minimum. The probability distributions and their characteristics are readily obtained from the values of the members of the pool. Convergence to a scaling regime is obtained within a few iterations. A summary of our results is shown in the table (see figures 5–8 for results for model E). The general result is that ω varies continuously as a function of α. For model E, $D_f = 1$ asymptotically for all $\alpha < \infty$ whereas $\alpha = \infty$ leads to $D_f = 1.15 \pm 0.003$. However, the region $\alpha < \infty$ corresponds to a regime of an α-independent self-affine geometry ($D_f = 1$). Note that non-trivial fractal behaviour of the optimal path is only possible for model E, since for the other models all distinct paths have exactly the same length and thus allow only one geometry.

Another interesting point of the model on a hierarchical lattice stems from the fact that it allows one to test some interesting issues relevant to the strong disorder regime itself. A key open problem is whether the D_f in the fractal regime is an independent exponent or related to any of the dimensions arising in the standard percolation problem. One of the candidates is the hull fractal dimension, which in 3D has a value very close to the numerically measured [9] fractal dimension of the optimal surface in the strong disorder limit.

A simple analytic estimate for D_f in the $\alpha = \infty$ limit for model E can be obtained by approximating the large length scale behaviour to be the same as that of a single unit of five bonds. This amounts to an assumption that the probabilities of the relative arrangements of the strengths of bonds within a given cell are not altered by the condition that the

cell considered is one through which the optimal path has to pass. On carrying out an exact enumeration of all possible rank orderings of the five bonds, one readily obtains $D_f = D_\infty = \ln(\frac{34}{15})/\ln 2 \approx 1.18$ which is close but not equal to the numerical estimate. Strikingly, by similar methods, the fractal dimension of the percolating cluster hull can also be shown to be D_∞ as well. This is done by treating the bond percolation problem. Denote the probability of bond occupation by p and performing the RG transformation, we find $p' = 2p^5 - 5p^4 + 2p^3 + 2p^2$ with the critical fixed point value $p_c = p^* = \frac{1}{2}$. We set $p = p_c$ on an infinite lattice and consider the number of hull bonds in such cells that rescale to just one hull bond to obtain our estimate. To our knowledge, this is the first determination of the hull dimension on a hierarchical lattice or in the context of real-space renormalization-group methods. While this coincidence is quite remarkable, it is possible that the two dimensionalities are generally not the same, since the optimal path D_f calculated numerically for both triangular and square lattices in $D = 2$ is smaller than that of the hull of the percolation cluster.

We now turn to an analytic calculation of ω in the strong disorder limit ($\alpha = \infty$) on hierarchical lattices. We begin by noting that on the lattices considered, directed percolation and ordinary percolation are one and the same. According to the Roux–Zhang argument [14], the energy (or equivalently the travel time) associated with the optimal path in the directed case is exactly the threshold for directed percolation, p_c, in the infinite-size limit. The energy fluctuations for finite size L, ΔE, scales as

$$\Delta E \sim |p - p_c| \sim \xi^{-1/\nu_\parallel} \sim L^{-1/\nu_\parallel} \tag{4}$$

where ξ is the correlation length. We thus obtain

$$\omega(\alpha = \infty) = -\frac{1}{\nu_\parallel}. \tag{5}$$

In order to obtain $\nu_\parallel$, we consider the real-space renormalization-group transformation $p' = R_b(p)$, where b is the scale factor. The fixed point $p^* = R_b(p^*)$ represents the energy of the optimal path whereas

$$\frac{1}{\nu_\parallel} = \frac{\ln \partial p'/\partial p}{\ln b}\bigg|_{p=p^*}. \tag{6}$$

The values of $-1/\nu_\parallel$ are presented in the table and are in excellent agreement with the numerical values of $\omega(\alpha = \infty)$.

In order to assess the generality of our results we have also studied the scale-dependent distribution (1) on the hierarchical lattice E with $\alpha = 1$. As on the square lattice, we find that for any $c > 1$, the interface crosses over from self-affine to fractal (figure 9). We have also studied the model defined by (2) in the context of directed polymers in a random medium using a transfer matrix method and the random-bond Ising model on a square lattice. In both cases we find an ω that depends on α. Indeed, assuming that the distribution of energies in the $\alpha = 1$ case is Gaussian with a mean value that scales as L^{χ_m} and a width scaling as L^{χ_w}, it is straightforward to show that

$$\omega(\alpha) = \chi_w - \chi_m(1 - 1/\alpha) \tag{7}$$

with $\chi_m = 1$ and $\chi_w = \omega(\alpha = 1)$. This result is valid for any $\alpha < \infty$ and exhibits a discontinuity at $\alpha = \infty$.

In summary, we have presented two different model calculations of the crossover from the weak-disorder self-affine optimal path to the strong disorder fractal optimal path. In the ferromagnetic domain wall problem on a square lattice, the width of the distribution was linked to the size of the system. The asymptotic geometry of the wall is modified

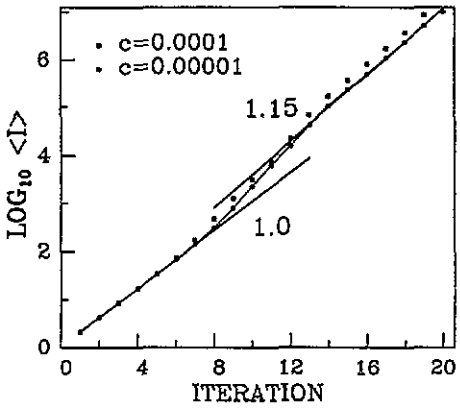

Figure 9. Variation of the average path length for the hierarchical lattice E in the case when the couplings (bond energies) correspond to (1). The values of the c-parameter are indicated. The pool consists of 500 000 bonds. For the data point corresponding to N_i iterations, the couplings in the initial pool are given by yc^z, where $z = 5^{N_i}(y - 0.5)$ and y is a random number between 0 and 1. The initial bond lengths are all equal to 1. The effective fractal dimensionalities D_f are indicated.

abruptly in that case from self-affine to fractal as soon as the control parameter $c > 1$. In the case of the hierarchical lattices, the path remains self-affine for a whole range of values of the parameter α. The geometry of the optimal paths studied is directly linked to the experimentally accessible scale-dependent tortuosity of disordered systems. Our studies address the crossover of the critical behaviour on turning from a regular cost space to an ultrametric one.

Acknowledgments

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